

## Smeared BTZ black hole from space noncommutativity

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**ABSTRACT:** We study a novel phenomena of smearing of black hole horizons from the effect of space noncommutativity. We present an explicit example in  $AdS_3$  space, using the Chern-Simons formulation of gravity. This produces a smeared BTZ black hole which goes beyond the classical spacetime unexpectedly and there is *no* reality problem in our approach with the gauge group  $U(1,1) \times U(1,1)$ . The horizons are smeared, due to a splitting of the Killing horizon and the apparent horizon, and there is a metric signature change to Euclidean in the smeared region. The inner boundary of the smeared region acts as a trapped surface for timelike particles but the outer as a classical barrier for ingoing particles. The lightlike signals can escape from or reach the smeared region in a *finite* time, which indicates that *the black hole is not so dark, even classically*. In addition, it is remarked that the Hawking temperature can *not* be defined by the regularity in the Euclidean geometry except in the non-rotating case, and the origin can be smeared by a *new* (apparent) horizon.

**KEYWORDS:** Models of Quantum Gravity, Chern-Simons Theories, Non-Commutative Geometry, Black Holes.

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**1. Introduction**

It is known that noncommutative field theories [1, 2] modify the short distance behaviors of conventional theories. For example, the noncommutativities can smooth out a singularity of classical solutions in the conventional theories [3]. However, similar modifications in gravity theories are not studied much, though there are several formulations of noncommutative gravities and speculations. For example, one can generally expect a “smearing” of the horizons, which are sharply defined in the conventional spacetime. From the commutation relations

$$[x^i, x^j] = i\theta^{ij}, \tag{1.1}$$

where  $\theta^{ij}$  is an antisymmetric constant of dimension  $length^2$ , the precise location of a horizon is limited by the uncertainty relations  $\Delta x^i \Delta x^j \geq |\theta^{ij}|/2$ . But, there has been no *explicit* demonstration of existence of the smearing region with unexpected space-time structure.

In this paper, we investigate the modifications of the BTZ black hole in three-dimensional anti-de Sitter ( $AdS_3$ ) space with a polar form of the commutation relation

$$[r, \phi] = i\hat{\theta} \tag{1.2}$$

for a spherically symmetric case ( $\theta^{r\phi} \equiv \hat{\theta}$ ),<sup>1</sup> using the Chern-Simons formulation of gravity. This is the first explicit example of the novel phenomena of the horizon smearing which goes beyond the classical spacetime unexpectedly, and *without* the reality problem by

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<sup>1</sup>This commutation relations differ from the Cartesian ones with a constant  $\theta^{ij}$  [4] since it corresponds to a non-constant  $\theta^{ij} = r\hat{\theta}\epsilon^{ij}$  in (1.1). But, the Moyal product [1] can be still consistently defined in the polar coordinate with a constant  $\theta^{r\phi}$ . And we expect no big difference in the *qualitative* physics near the horizon since we are considering physics at  $r \neq 0$ .

considering the gauge group  $U(1, 1) \times U(1, 1)$ . It is found that the event horizon is *smear*ed by the splitting of the Killing horizon and the apparent horizon with the thickness of the order  $\hat{\theta}$  and the smeared horizon acts as a classical barrier for particles. The time duration of light signals escaping from and reaching the smeared horizon region from outside is demonstrated to be finite. The physical effect of the metric signature change inside the smeared region is considered. In addition, it is remarked that the Hawking temperature from the regularity in the Euclidean geometry is *not* applicable, and the naked conical singularity at the origin in  $AdS_3$  or negative mass solutions, generally, is smeared due to the presence of a “new” apparent horizon near the origin.

## 2. Three-dimensional noncommutative gravity from Chern-Simons formulation

It is well known that, in three-dimensional space-time, conventional gravity theory can be formulated as a Chern-Simons theory [5]. This provides a novel way to define the noncommutative gravity theory [6]. An explicit solution in the noncommutative gravity can be obtained from the corresponding known solution in the commutative gravity, via the Seiberg-Witten map [7].

The (2+1)-dimensional noncommutative gravity with the negative cosmological constant  $\Lambda = -1/l^2$  is defined by the  $U(1, 1) \times U(1, 1)$  Chern-Simons theory, up to surface terms,

$$I_g[\hat{A}] = \frac{l}{16\pi G} \int Tr \left( \hat{A}^+ \star d\hat{A}^+ + \frac{2}{3} \hat{A}^+ \star \hat{A}^+ \star \hat{A}^+ \right) - \left( \hat{A}^+ \leftrightarrow \hat{A}^- \right). \quad (2.1)$$

(Here, the wedge symbol has been omitted.) The connections are given by<sup>2</sup>

$$\hat{A}^\pm = (\hat{\omega}^a \pm \hat{e}^a/l) t_a + i\hat{c}^\pm \mathbf{1} \quad (2.2)$$

with the triads and the  $SU(1, 1)$  spin connections  $\hat{e}^a = \hat{e}_\mu^a dx^\mu$ ,  $\hat{\omega}_\mu^a = (1/2)\epsilon^{abc}\hat{\omega}_{\mu bc}dx^\mu$  ( $a = 0, 1, 2$ ), respectively, and the  $U(1)$  connections  $c^\pm$  which make the group closed with respects to the Moyal  $\star$  product [1, 6]

$$\star = \exp \left[ \frac{i}{2} \hat{\theta} (\overleftarrow{\partial}_r \overrightarrow{\partial}_\phi - \overleftarrow{\partial}_\phi \overrightarrow{\partial}_r) \right]. \quad (2.3)$$

Here, it is important to note that, in the commutative limit, the theory does not depend on the metric, i.e., the choice of the coordinates [8].<sup>3</sup> In the polar coordinates  $\mu = (t, r, \phi)$ , for example, the (commutative) action takes the form [9]:  $I_g[A] =$

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<sup>2</sup>We take the  $SU(1, 1)$  bases  $t_0 = \sigma_2/2, t_1 = i\sigma_3/2, t_2 = \sigma_1/2$  such that  $[t_a, t_b] = -\epsilon_{ab}^c t_c, Tr(t_a t_b) = (1/2)\tilde{\eta}_{ab}$  with  $\epsilon_{012} = 1$  and  $\tilde{\eta}_{ab} = diag(+1, -1, +1)$ .

<sup>3</sup>The metric dependence in the Cartesian measure of integration,  $d^3x$ , under the general coordinate transformations,  $d^3x' = d^3x/\sqrt{g}$ , is canceled by that of the Levi-Civita tensor density  $\epsilon^{\alpha\beta\gamma} = \sqrt{g}\partial_\mu x'^\alpha \partial_\nu x'^\beta \partial_\rho x'^\gamma \epsilon^{\mu\nu\rho}$  in the Chern-Simons 3-form  $d^3x \epsilon^{\mu\nu\rho} Tr[A_\mu \partial_\nu A_\rho + (2/3)A_\mu A_\nu A_\rho]$ .

$(l/16\pi G) \int dt dr d\phi \text{Tr}(-A_r^+ \partial_t A_\phi^+ + A_\phi^+ \partial_t A_r^+ + 2A_t^+ F_{r\phi}^+) - (A_\mu^+ \leftrightarrow A_\mu^-)$ . We consider the Moyal  $\star$  deformation of this polar action as follows:

$$I_g[\hat{\mathcal{A}}] = \frac{l}{16\pi G} \int dt dr d\phi \text{Tr} \left( -\hat{\mathcal{A}}_r^+ \star \partial_t \hat{\mathcal{A}}_\phi^+ + \hat{\mathcal{A}}_\phi^+ \star \partial_t \hat{\mathcal{A}}_r^+ + 2\hat{\mathcal{A}}_t^+ \star \hat{\mathcal{F}}_{r\phi}^+ \right) - \left( \hat{\mathcal{A}}_\mu^+ \leftrightarrow \hat{\mathcal{A}}_\mu^- \right). \tag{2.4}$$

(We define  $\hat{\mathcal{F}}_{\mu\nu}^\pm = \partial_\mu \hat{\mathcal{A}}_\nu^\pm - \hat{\mathcal{A}}_\mu^\pm \star \hat{\mathcal{A}}_\nu^\pm - (\mu \leftrightarrow \nu)$ , in a covariant way.) We note that the noncommutative action has the measure of integration  $dt dr d\phi$  and so the Moyal product (2.3) is well defined, as in the usual Cartesian coordinates [7]. There are some differences in the global properties of the coordinates, due to the range of the coordinates  $(0, \infty) \times (0, 2\pi)$  and so there will be some appropriate boundary conditions on the allowed functions (see footnote 4, for example). However all the standard ‘‘local’’ (i.e., ignoring boundary conditions) formula in Cartesian coordinates, like equations of motion and the Seiberg-Witten map, will work also here [6].

The equations of motion are given by

$$d\hat{\mathcal{A}}^\pm + \hat{\mathcal{A}}^\pm \star \hat{\mathcal{A}}^\pm = 0 \tag{2.5}$$

which are not easy to solve directly. But using the Seiberg-Witten map [10], which transforms the noncommutative Chern-Simons action into the commutative one, or vice versa,<sup>4</sup> without additional action [7], any solution  $\mathcal{A}^\pm$  of the commutative equations  $d\mathcal{A}^\pm + \mathcal{A}^\pm \mathcal{A}^\pm = 0$  can be mapped into the corresponding  $\hat{\mathcal{A}}^\pm$  of the noncommutative equations (2.5) [11]:

$$\hat{\mathcal{A}}_\mu^\pm(\theta) = \mathcal{A}_\mu^\pm + (i/4)\theta^{\alpha\beta}[\mathcal{A}_\alpha^\pm, \partial_\beta \mathcal{A}_\mu^\pm + \mathcal{F}_{\beta\mu}^\pm]_+ + \mathcal{O}(\theta^2). \tag{2.6}$$

From the obtained solution  $\hat{\mathcal{A}}^\pm$ , one can compute  $\hat{\omega}$  and  $\hat{e}$  which describe the noncommutative gravity.

### 3. BTZ black hole solution with U(1) fluxes in U(1, 1) × U(1, 1) Chern-Simons gravity

In the commutative limit, the equations of motion reduce to the two sets of *decoupled* equations

$$d\omega + \omega\omega - (1/l^2)ee = 0, \quad de + \omega e + e\omega = 0, \tag{3.1}$$

$$dc^\pm = 0. \tag{3.2}$$

We generalize these to the case with the non-vanishing U(1) fluxes  $dc^\pm = f^\pm$  by adding an additional term  $-2 \int \text{Tr}(f\mathcal{A})$  to the Chern-Simons gravity action. This modifications do not change the conventional gravity equations (3.1), since these are decoupled from the U(1)-parts (3.2).

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<sup>4</sup>In this mapping, there are several boundary terms which do not vanish for an arbitrary choice of  $\theta^{ij}$ , generally. But, these terms vanish for our choice (1.2) and the commutative solutions  $\mathcal{A}^\pm$  which decrease rapidly for large  $r$ .

In the noncommutative case, we have the additional noncommutative action  $-2 \int Tr(\hat{f} \star \hat{A})$  [12]. One can confirm that this term is invariant under the Seiberg-Witten map, for the appropriate fluxes  $\hat{f}$  which decrease rapidly for large  $r$  with our choice (1.2). In consequence, the solution  $\hat{A}$  of the generalized theory can be simply obtained by the same mapping (2.6) from the corresponding commutative solution  $\mathcal{A}$ . The U(1) fields are not decoupled anymore and have non-trivial effects on the commutative gravity solutions.

To proceed, we consider the Aharonov-Bohm type U(1) potentials

$$c^\pm = \Phi^\pm d\phi \tag{3.3}$$

which give the fluxes inside the horizons,  $f^\pm = 2\pi\Phi^\pm\delta^2(\mathbf{x})drd\phi$ . For the (commutative) gravity solution, we consider the BTZ black hole given by [13]

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2(d\phi + N^\phi dt)^2 \tag{3.4}$$

with  $N^2 = (r^2 - r_+^2)(r^2 - r_-^2)/l^2 r^2$ ,  $N^\phi = -r_+ r_- / l r^2$ . Here,  $r_+$  and  $r_-$  denote the outer and inner horizons, respectively. The  $SU(1, 1) \times SU(1, 1)$  1-form gauge connections  $\mathbf{A}^\pm = (\omega^a \pm e^a/l)t_a$  are given by [14]

$$\mathbf{A}^\pm = \frac{1}{2} \begin{pmatrix} \pm i(l/\nu)d\mu & z_\pm(\nu \mp i\mu)(dt/l \mp d\phi) \\ z_\pm(\nu \pm i\mu)(dt/l \mp d\phi) & \mp i(l/\nu)d\mu \end{pmatrix}, \tag{3.5}$$

where  $\nu^2(r) = (r^2 - r_-^2)/(r_+^2 - r_-^2)$ ,  $\mu^2 = \nu^2 - 1$ , and  $z_\pm = (r_+ \pm r_-)/l$ .

For the noncommutativity relations (1.2) with  $\hat{\theta}$  rescaled as  $\hat{\theta} = l\theta$  and others=0, the solutions  $\hat{A}^\pm$  in the noncommutative gravity (2.4) are obtained, via the Seiberg-Witten map,

$$\begin{aligned} \hat{A}_t^\pm &= i(l\theta/4)Tr(A_\phi^\pm \partial_r A_t^\pm) \mathbf{1} + [1 + (l\theta/2)c_\phi^\pm \partial_r] A_t^\pm, \\ \hat{A}_r^\pm &= i(l\theta/4)Tr(A_\phi^\pm \partial_r A_r^\pm) \mathbf{1} + [1 + (l\theta/2)\{(\partial_r c_\phi^\pm) + c_\phi^\pm \partial_r\}] A_r^\pm, \\ \hat{A}_\phi^\pm &= [1 + (l\theta/2)(\partial_r c_\phi^\pm)] A_\phi^\pm, \end{aligned} \tag{3.6}$$

(note that  $\mathcal{A}_{t,r}^\pm = A_{t,r}^\pm$ ,  $\mathcal{A}_\phi^\pm = c_\phi^\pm + A_\phi^\pm$ ), neglecting the higher order terms of  $\mathcal{O}(\theta^2)$ . The metric of the noncommutative gravity is defined<sup>5</sup> as  $d\hat{s}^2 = \eta_{ab}(\hat{e}_\mu^a \star \hat{e}_\nu^b)dx^\mu dx^\nu$  with  $\eta_{ab} = \text{diag}(-1, +1, +1)$  and given by

$$d\hat{s}^2 = -f^2 dt^2 + \hat{N}^{-2} dr^2 + r^2(d\phi + N^\phi dt)^2 + \mathcal{O}(\theta^2), \tag{3.7}$$

where

$$\hat{N}^2 = \frac{r^2}{l^2} + \theta c_\phi \frac{r}{l} - \frac{(r_+^2 + r_-^2)}{l^2} + \frac{r_+^2 r_-^2}{l^2 r^2} - \theta c_\phi \frac{r_+^2 r_-^2}{l r^3}, \tag{3.8}$$

$$f^2 = \hat{N}^2 + \theta c_\phi \frac{r_+^2 r_-^2}{l r^3}. \tag{3.9}$$

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<sup>5</sup>Here, the metric has no “reality problem” since the equality  $\hat{e}_\mu \star \hat{e}_\nu = \hat{e}_\mu \hat{e}_\nu$  holds in our metric (3.4) due to the commutation relations (1.2). This is a unique feature of our approach and in sharply contrast to the previous approaches [15] where the metric becomes *complex* generally and some truncations are needed to get a real metric.

Here, we consider  $c_\phi^+ = c_\phi^-$  for simplicity and omit the singular term at the origin  $-2\pi l\theta N^2 \Phi \delta^2(\mathbf{x})$  in  $\hat{N}^2$  since we are considering physics at  $r \neq 0$ . Note that, in this case, the noncommutative spacetime satisfies the same *gravity* equations of motion as in the commutative limit, i.e.,  $d\hat{\omega} + \hat{\omega}\hat{\omega} - (1/l^2)\hat{e}\hat{e} = 0$ ,  $d\hat{e} + \hat{\omega}\hat{e} + \hat{e}\hat{\omega} = 0$ , with *no* Moyal products, from (2.5) and the triviality of bi-products in our solution (3.6), i.e.,  $\hat{\omega} \star \hat{\omega} = \hat{\omega}\hat{\omega}$ ,  $\hat{\omega} \star \hat{e} = \hat{\omega}\hat{e}$ , etc. In other words, the (first-order) noncommutative solution (3.6) has the same noncommutative curvature  $\hat{R} \equiv d\hat{\omega} + \hat{\omega}\hat{\omega} = (1/l^2)\hat{e}\hat{e}$  and (zero) torsion  $\hat{T} \equiv d\hat{e} + \hat{\omega}\hat{e} + \hat{e}\hat{\omega} = 0$ , as in the conventional BTZ black hole spacetime, outside the point flux [14].<sup>6</sup>

#### 4. Properties of the smeared black hole

The noncommutative black hole solution has several remarkable properties which go beyond the classical geometry, unexpectedly.

1. *There is a splitting of the apparent horizon and the Killing horizon:* The apparent horizon is defined as a null hypersurface  $g^{\mu\nu}(\partial_\mu r)(\partial_\nu r) = \hat{N}^2 = 0$ , whereas the Killing horizon as the surface where the norm of the Killing vector  $\chi = \partial_t + \Omega_H \partial_\phi$  vanishes, i.e.,  $\chi^2 = g_{tt} - (g_{t\phi})^2/g_{\phi\phi} = -f^2 = 0$  with the angular velocity of the horizon  $\Omega_H = -(g_{t\phi}/g_{\phi\phi})|_H$ .

We first note that  $\hat{N}^2 = [x^5 + \theta c_\phi x^4 - (x_+^2 + x_-^2)x^3 + (x_+^2 x_-^2)x - \theta c_\phi x_+^2 x_-^2]/x^3 = 0$  ( $x \neq 0$ ,  $x \equiv r/l$ ) has the outer/inner (apparent) horizons at

$$\hat{r}_\pm = r_\pm - l\theta c_\phi/2. \tag{4.1}$$

The apparent horizons are *equally* shifted by the small amount  $-l\theta c_\phi/2$ . On the other hand, the outer/inner Killing horizons can be obtained from  $f^2 = [x^4 + \theta c_\phi x^3 - (x_+^2 + x_-^2)x^2 + x_+^2 x_-^2]/x^2 = 0$ ,

$$\tilde{r}_\pm = r_\pm - (l\theta c_\phi/2)(1 - r_\mp^2/r_\pm^2)^{-1}. \tag{4.2}$$

This shows that Killing horizons are *not* equally shifted so that the apparent and Killing horizons do not coincide each other in general, except in the *non-rotating* BTZ ( $r_- = 0$ ).<sup>7</sup> It is noted that the solution (4.2) of the Killing horizons are not valid near the extremal commutative black holes with  $r_+ = r_-$  where the higher order corrections are needed in contrast to the apparent horizons (4.1).

2. *The event horizon becomes “smeared”, due to the splitting of the Killing and apparent horizons:* To see this, let us consider the metric

$$d\hat{s}^2 = -f^2 dt^2 + \hat{N}^{-2} dr^2 + r^2(d\tilde{\phi} + \tilde{N}^\phi dt)^2, \tag{4.3}$$

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<sup>6</sup>This may be compared with a BTZ solution in the presence of higher derivative terms, like the gravitational Chern-Simons action term. There the BTZ solution satisfies, trivially, the Einstein’s equations of motion, with no higher derivative contributions. However, this does not mean a triviality of the solution since its physical parameters, like the ADM mass and angular momentum are significantly deformed from the conventional ones [16].

<sup>7</sup>However, the inner horizons  $\hat{r}_-, \tilde{r}_-$  are absent in this case, though not manifest in (4.1) -(4.2).

such as  $\tilde{N}^\phi = 0$  either at the Killing horizons  $\tilde{r}_+$  or the apparent horizon  $\hat{r}_+$ , for an appropriate choice of the *co-rotating* frame. Then, the radial null geodesics are given by

$$dr/dt = \pm \sqrt{f^2 \hat{N}^2}, \quad (4.4)$$

with the upper (lower) sign corresponding to outgoing (ingoing) geodesics.

Near the Killing horizon  $\tilde{r}_+$ , the radial null geodesics for  $r > \tilde{r}_+$  are given by

$$dr/dt = \pm \sqrt{2\tilde{\kappa}(-\theta_{c_\phi})r_+^2 r_-^2 / l\tilde{r}_+^3} \sqrt{r - \tilde{r}_+}, \quad (4.5)$$

where  $\tilde{\kappa}$  corresponds to the surface gravity in the usual context ( $f^2 \approx 2\tilde{\kappa}(r - \tilde{r}_+)$ ):

$$\begin{aligned} \tilde{\kappa} &= (1/2)(\partial f^2 / \partial r)|_{\tilde{r}_+} \\ &= (\tilde{r}_+^4 - r_+^2 r_-^2) / (l^2 \tilde{r}_+^3) + \theta_{c_\phi} / 2l. \end{aligned} \quad (4.6)$$

For the non-negative  $\tilde{\kappa}$ , which is always the case when  $\theta$  is not so large, the outgoing as well as the ingoing geodesics are “classically” allowed only if the velocity (4.5) is real, i.e.,  $\theta_{c_\phi} < 0$ , or  $r_+ < \hat{r}_+ < \tilde{r}_+$ .

Even though the light cones close up as we approach the horizon  $\tilde{r}_+$ , which signals usually that the time coordinate  $t$  is badly defined near the horizon, it is remarkable that the light signals can escape from and reach the horizon  $\tilde{r}_+$  in a *finite* time

$$\tilde{t} \approx [\tilde{\kappa}(-\theta_{c_\phi})r_+^2 r_-^2 / 2l\tilde{r}_+^3]^{-1/2} \sqrt{r - \tilde{r}_+}. \quad (4.7)$$

This is in contrast to the conventional commutative case,<sup>8</sup> where one needs an infinite time  $t \sim \ln(r - r_+)$  to escape from the horizon  $r_+$  though a finite “proper” time to reach the horizon [18]. It seems that the singular behavior of the time coordinate  $t$  near the horizon is “moderated” by the noncommutativity effect. Thus, the horizon is smeared and *not so “dark”, even classically!*

The same is true when  $\theta_{c_\phi} > 0$ , or  $\tilde{r}_+ < \hat{r}_+ < r_+$ . In this case, the radial null geodesics for  $r > \hat{r}_+$  near the apparent horizon  $\hat{r}_+$  is given by

$$dr/dt = \pm \sqrt{2\hat{\kappa}(\theta_{c_\phi})r_+^2 r_-^2 / l\hat{r}_+^3} \sqrt{r - \hat{r}_+}, \quad (4.8)$$

where  $\hat{N}^2 \approx 2\hat{\kappa}(r - \hat{r}_+)$  and  $f^2 \approx (2\hat{\kappa} - 3\theta_{c_\phi}r_+^2 r_-^2 / l\hat{r}_+^4)(r - \hat{r}_+) + \theta_{c_\phi}r_+^2 r_-^2 / l\hat{r}_+^3$  are used with

$$\begin{aligned} \hat{\kappa} &= (1/2)(\partial \hat{N}^2 / \partial r)|_{\hat{r}_+} \\ &= (\hat{r}_+^4 - r_+^2 r_-^2) / (l^2 \hat{r}_+^3) + \theta_{c_\phi} / 2l + 3\theta_{c_\phi}r_+^2 r_-^2 / (2l\hat{r}_+^4). \end{aligned} \quad (4.9)$$

The geodesics show that the escaping time from (or approaching time to) the horizon  $\hat{r}_+$  is finite

$$\hat{t} \approx [\hat{\kappa}(\theta_{c_\phi})r_+^2 r_-^2 / 2l\hat{r}_+^3]^{-1/2} \sqrt{r - \hat{r}_+}. \quad (4.10)$$

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<sup>8</sup>Note that this behavior can *not* be directly obtained by setting  $\theta_{c_\phi} = 0$  in (4.7) since we must consider  $(r - r_+)$ -term again which now dominates  $\theta_{c_\phi}$ -term, in that case.

3. *The smeared horizon region behaves as a barrier for particles and waves:* This comes from the fact that  $f^2 \hat{N}^2 < 0$  in the region between  $\tilde{r}_+$  and  $\hat{r}_+$  and one has the *imaginary* radial velocities for the (radial) null geodesics (4.4).<sup>9</sup> This is the consequence of the fact that there is a “signature change”<sup>10</sup> to Euclidean (+++), in the smeared region, i.e., there is “no-time” and there are no light cones for the radial motions when  $\theta_{c_\phi} < 0$ . In this sense, the outer (horizon) boundary becomes *hard* to penetrate for particles, compared to the conventional event horizon. Nevertheless, light wave<sup>11</sup> may tunnel the smeared horizon region when its wavelength is greater than the thickness of the region. (Similar thing happens when  $\theta_{c_\phi} > 0$ . In this case, we have (pseudo) Euclidean geometry with the signature (− − +), i.e., there are “two-times” in the smeared region, and time and radial coordinates change the role.)

However, particles or waves inside the inner boundary can not escape from the black hole since the light cone structures in that region are the same as in the interior region of the event horizon of the commutative case. So, the inner boundary of the smeared region is the trapped surface. The usual Hawking radiation would be generated near the inner boundary since one of the pair-created particles can be trapped. On the other hand, the pair-created particles near the outer boundary always recombine, due to the absence of the trapping. Thus, our result seems to favor the tunneling picture of the Hawking radiation by Parikh and Wilczek [18]. Further studies are needed in this direction.

4. *The Hawking temperature defined by the periodicity in the Euclidean time is not applicable:* To see this, let us consider the metric (4.3) near the Killing horizon  $\tilde{r}_+$ . Following the usual approach, we put the Euclidean time  $\tau = -it$  and we get

$$d\hat{s}^2 \approx \tilde{\kappa}^2 \eta^2 d\tau^2 + \left[ \left( 1 + \frac{3\theta_{c_\phi} r_-^2}{2lr_+^2 \tilde{\kappa}} \right) - \frac{\theta_{c_\phi} r_-^2}{lr_+ \tilde{\kappa}^2 \eta^2} \right]^{-1} d\eta^2 + (\tilde{r}_+ + \tilde{\kappa} \eta^2 / 2)^2 (d\tilde{\phi})^2, \quad (4.11)$$

where  $\eta = \sqrt{2(r - \tilde{r}_+)}$ . From the regularity of the  $d\tau^2$ -part, one obtains the periodicity  $\beta = 2\pi/\tilde{\kappa}$  which would give the conventional Hawking temperature  $T_H = \tilde{\kappa}/2\pi$  [19]. For the non-rotating case ( $r_- = 0$ ), the system is quite normal and the Killing and apparent horizons coincide, except the shift of the horizon in (4.1). In this case, the Hawking temperature can be defined as usual with a constant shift:<sup>12</sup>  $\hat{T}_H = (\hat{r}_+ + l\theta_{c_\phi}/2)/(2\pi l^2)$ .

However, the (near horizon) geometry spoils the regularity in general since, as  $\eta \rightarrow 0$ , the radial coordinate  $\eta$  becomes time-like and the Euclidean procedure itself becomes

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<sup>9</sup>This can be also observed in (4.5) and (4.8).

<sup>10</sup>This seems to reflect the *quantum gravity* nature of the noncommutative geometry, as is in the beginning of the Universe [17].

<sup>11</sup>Particles may tunnel *quantum mechanically*, by their wave nature also.

<sup>12</sup>If one considers the Bekenstein-Hawking entropy [20],  $\hat{S}_{BH} = 2\pi\hat{r}_+$  with respect to the horizon  $\hat{r}_+$ , one would have the first law of thermodynamics with the black hole mass  $\hat{M} = (\hat{r}_+^2 + l\theta_{c_\phi}\hat{r}_+)/2l^2$ , up to some additive constant terms. But, it is not clear whether the Bekenstein’s area law is satisfied even in the noncommutative geometry, which can be regarded as the higher derivative gravities [16].



invalid. Thus, the conventional way of defining the temperature is not valid anymore and this might be a general phenomena for the noncommutative geometry with the smeared horizons. Somehow we suspect that  $T_H = \tilde{\kappa}/2\pi$ , which converges to the usual Hawking temperature  $T_H = \kappa_+/2\pi$  for the commutative case, can represent the characteristic of the thermodynamical temperature of the smeared systems. The details of this definition are beyond the scope of this paper.

5. *The origin  $r = 0$  is also smeared by a “new” horizon for  $\theta c_\phi > 0$ :* There is a third solution of  $\hat{N}^2 = 0$  at

$$\hat{r}_{--} = l\theta c_\phi \tag{4.12}$$

when  $\theta c_\phi > 0$ . This provides a new “apparent horizon”,<sup>13</sup> inside the inner horizon, encircling the origin. The interior region of  $\hat{r}_{--}$  has two-times as in the region between  $\tilde{r}_+$  and  $\hat{r}_+$ .

In the pure  $AdS_3$  solutions or the *negative mass* black holes, generally, which can be obtained by considering  $r_\pm^2 \rightarrow -r_\pm^2$  in the black hole solution (3.4), there is no event horizon and there appears a naked conical singularity at the origin. The conical singularity may be smeared by this new horizon as the results of the spatial noncommutativity

It is also noted that the appearance of the new horizon near the origin depends on the sign of  $\theta c_\phi$ , which is analogous to the sign dependence on the existence of soliton solutions, “fluxons” in the field theories [3].

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<sup>13</sup>Similar situation also occurs in AdS black holes with self-interacting scalar hairs [21].

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